Sample $\LaTeX$ Document

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1. Use the formal definition of the limit of a function at a point to prove that the following holds:

$$\lim_{x\to 4} x^2 + x - 5$$

Proof. Fix an arbitrary $\epsilon > 0$.

We wish to determine a $\delta > 0$ such that when $0 < |x-4| < \delta$, it must be true that $|f(x) - 15| < \epsilon$.

Choose $\delta = \min\left\{1, \dfrac{\epsilon}{10}\right\}$.

Now, suppose that $0 < |x-4| < \delta$. Then,

$$\begin{aligned}
|f(x) - 15| &= |(x^2 + x - 5) - 15|, \text{ by the definition of } f, \\
&= |x^2 + x - 15| \\
&= |(x-4)(x+5)| \\
&= |x-4||x+5|, \text{ by properties of absolute value,} \\
&< \delta \cdot |x+5|, \text{ by the assumption } |x-4| < \delta, \\
&\leq \frac{\epsilon}{10}|x+5|, \text{ since } \delta \leq \frac{\epsilon}{10}, \\
&= \frac{\epsilon}{10}|(x-4)+9| \\
&\leq \frac{\epsilon}{10}\left(|x-4| + |9|\right), \text{ by properties of absolute value,} \\
&< \frac{\epsilon}{10}\left(\delta + 9\right), \text{ since } |x-4| < \delta, \\
&\leq \frac{\epsilon}{10}(1+9), \text{ since } \delta \leq 1, \\
&= \left(\frac{\epsilon}{10}\right)(10) = \epsilon
\end{aligned}$$

Scratch Work

$$\begin{aligned}
|f(x) - 15| &< \epsilon \\
|(x^2 + x - 5) - (15)| &< \epsilon \\
|x^2 + x - 20| &< \epsilon \\
|(x+5)(x-4)| &< \epsilon \\
|(x-4)| \cdot |(x+5)| &< \epsilon \\
|x-4| &< \frac{\epsilon}{|x+5|}
\end{aligned}$$

$$\begin{aligned}
\delta = 1 \implies |x-4| &< 1 \\
-1 < x - 4 &< 1 \\
8 < x + 5 &< 10 \\
|x+5| &< 10
\end{aligned}$$

All together, this shows that for any $\epsilon > 0$, if we choose $\delta = \min\left\{1, \dfrac{\epsilon}{10}\right\}$, then $0 \leq |x-4| < \delta$ implies that $|f(x) - 15| < \epsilon$. Thus, $\lim_{x\to 4} x^2 + x - 5 = 15$.

$\square$

1.5.15 Evaluate the given limits of the piecewise defined function f.

$$f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x < -1 \\ x^3 + 1 & \text{if} \quad -1 \le x \le 1 \\ x^2 + 1 & \text{if} \quad x > 1 \end{cases}$$

(a) $\lim_{x \to -1^-} f(x)$

Since we are evaluating the limit as x approaches -1 from the left, we need to consider the form of the function for values of x that are less than -1, $x^2 - 1$.

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} x^2 - 1 \\ &= (-1)^2 - 1, \text{ by Theorem 2,} \\ &= 0. \end{aligned}$$

(b) $\lim_{x \to -1^+} f(x)$

Since we are evaluating the limit as x approaches -1 from the right, we need to consider the form of the function for values of x that are greater than -1, $x^3 + 1$.

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} x^3 + 1 \\ &= (-1)^3 + 1, \text{ by Theorem 2,} \\ &= 0. \end{aligned}$$

(c) $\lim_{x \to -1} f(x)$

Since $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = 0$, $\lim_{x \to -1} f(x) = 0$ by Theorem 7.

(d) $f(-1)$

When $x = -1$, $f(x) = x^3 + 1$. So, $f(-1) = (-1)^3 + 1 = 0$.

(e) $\lim_{x \to 1^-} f(x)$

Since we are evaluating the limit as x approaches 1 from the left, we need to consider the form of the function for values of x that are less than (but near) 1, $x^3 + 1$.

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x^3 + 1 \\ &= (1)^3 + 1, \text{ by Theorem 2,} \\ &= 2. \end{aligned}$$

(f) $\lim_{x \to 1^+} f(x)$

Since we are evaluating the limit as x approaches 1 from the right, we need to consider the form of the function for values of x that are greater than (but near) 1, $x^2 + 1$.

$$\begin{aligned}\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^+} x^2 + 1 \\ &= (1)^2 + 1, \text{ by Theorem 2,} \\ &= 2.\end{aligned}$$

(g) $\lim_{x \to 1} f(x)$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$, $\lim_{x \to 1} f(x) = 2$ by Theorem 7.

(h) $f(1)$

When $x = 1$, $f(x) = x^3 + 1$. So, $f(1) = (1)^3 + 1 = 2$.

To help us visualize all of these limits, a graph of $y = f(x)$ is provided below.