The Schroeder-Bernstein Theorem

Let S and T be two sets such that f: S $\rightarrow$ T and g: T $\rightarrow$ S where f and g are both injective functions. Prove that there exists a bijective function h: S $\rightarrow$ T.

Proof. This theorem is very intuitive for the finite dimensional case. Pick any natural number n. Let n represent the cardinality (number of elements) of the set S. For an injection f to exist, the cardinality of set T must be $\geqslant$ the cardinality of set S. But for an injection g to exist, the cardinality of set S must be $\geqslant$ the cardinality of set T. Thus, $\mathbf{card}(S) = \mathbf{card}(T)$ and there exists a bijective function h from S to T.

For the infinite dimensional case, consider the following problem: **what if** $\exists\, x_0$ **s.t** $x_0 \in S$ **but** $x_0 \notin g(T)$**?** The definition of f guarantees that $f(x_0) \in T$. Hence, $g(f(x_0)) \in S$. Now, if $f(g(f(x_0))) = f(x_0)$ then $f(a) = f(b)$ for some $a \neq b$; thus, there is a contradiction because we know that f is injective. Conversely, what happens in case $f(g(f(x_0))) \neq f(x_0)$? We end up with a cycle of consecutive images of f and g. Somehow, we must show that even that cycle causes a contradiction that prevents the existence of a bijective function h! **We must formalize our intuition now.**

Let $X_0 = S \setminus g(T)$. Let the cycle of X_n be defined recursively by $X_n = g(f(X_{n-1}))$. Finally, let $X = \cup_{n \in N} X_n$ where $X \supset X_0 = S \setminus g(T)$. Define h: S $\rightarrow$ T by the following construction:

$$h(x) = \begin{cases} f(x), & \text{if } x \in X, \\ g^{-1}(x), & \text{if } x \notin X. \end{cases}$$

We claim that h is bijective.

h is injective: Let $x, x_0 \in S$ with $h(x) = h(x_0)$. If $x, x_0 \in X$ then $f(x) = h(x) = h(x_0) = f(x_0)$ and we're done because f is injective. If both $x, x_0 \notin X$, then $x = g(h(x)) = g(h(x_0)) = x_0$. However, when $x \in X$ and $x_0 \notin X$, we must approach this differently. We know that $h(x) = f(x)$. This means that $g^{-1}(x_0) = h(x_0) = h(x) = f(x)$ and composing with g means $g(f(x)) = x_0$. However, $g(f(x))$ must be in X, but x_0 was defined not to be in X - leading to a contradiction.

h is surjective: Let $y \in T$. We want to show that there exists an x such that $h(x) = y$. Let $x' = g(y)$. If $x' \notin X$, we have that $h(x') = g^{-1}(x') = y$. Let $x = x'$ and we are done. However, if $x' \in$ X, then $x' \in X_n$ for some n. $n \neq 0$ because x' is the image of some y under g. Let $x \in X_{n-1}$, be such that $x' = g(f(x))$. Then, we know that $g(f(x)) = x' = g(y)$. Hence, $f(x) = y$ which is exactly what we wanted because in this sub case $f(x) = h(x)$.

Phew! We first constructed h, a function from S to T, and now proved that it is bijective. $\square$