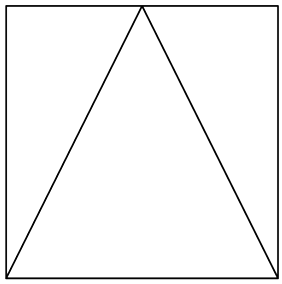
Solutions Document

PersonPsychopath's Mock AMC 10

1. We note that $\sqrt[3]{(x-2)^3} = \sqrt[3]{-125} \implies x-2=-5 \implies x=-3$. Thus, we have that $(x-3)^2 = (-3-3)^2 = \boxed{\textbf{(E) } 36}$.

2. The square and the triangle have the same base and height length, but since MCD is a triangle, we must divide by 2 to find its area. Thus, the ratio we are looking for is 1:2, which is $\boxed{\textbf{(D) } 1:2}$.



3. If Liam loses 2 bets, his money will be multiplied by $\frac{1}{2}$ 2 times, in essence

$$(\frac{1}{2})^2 = \frac{1}{4}$$

After he wins, his money will be multiplied by $\frac{3}{2}$ 2 times, or

$$(\frac{3}{2})^2 = \frac{9}{4}$$

Therefore:

$$400 \cdot \frac{1}{4} \cdot \frac{9}{4} = 400 \cdot \frac{9}{16} = \boxed{\textbf{(B) } 225}$$

4. We note that 147 is 4 mod 11 and that 258 is 5 mod 11. Multiplying, we get 20 mod 11. This reduces to $\boxed{\textbf{(E) } 9}$ mod 11.

5. The numbers 15, 24, 33, 42, and 51 have the property that the sum of their digits is 6. There are 5 of these. The numbers 16, 25, 34, 43, 52, and 61 have the property that the sum of their digits is 7. There are 6 of these. In total our answer is $\boxed{\textbf{(C) } 11}$.

6. Note that EC = 20, and DE = 5. Thus, by the Pythagorean Theorem, AE= BC= $\boxed{\textbf{(E) } 12}$.

7. We can reduce this to asking for how many 5-digit sequences (where the leading digit can be 0) exist such that it contains the block 56. Given 2 digits that have the 56 block, there are 10^2 to pick the other 2 digits. By

symmetry, there are $3 \times 100 = 300$ numbers. But don't forget that 0000 is not included in our list of positive integers from 1 - 10000 so our answer is actually $300 - 1 = \boxed{\textbf{(B) } 299}$.

8. Since this is an arithmetic sequence, and the sum is 180, then the middle term must be the average of the first and last term, so the middle term must be $\frac{180}{3} = 60$. We're also given the measure of an additional angle in the triangle: 25. Then the other term must be symmetric about 60, so $60 + (60 - 25) = \boxed{\textbf{(C) } 95}$.

9. The minimal possible sum is $1 + 2 + 3 + ... + 90$. The maximum is $10 + 11 + 12 + ... + 99$. So the difference is $(10 - 1) + (11 - 2) + ... + (99 - 90) = 9 \times 90 = \boxed{\textbf{(A) } 810}$.

10. Let's count what we don't want and then subtract that from the total number of ways to do this. The cases that break the rules are having all four doors the same color or having 3 out of 4 doors be the same color.

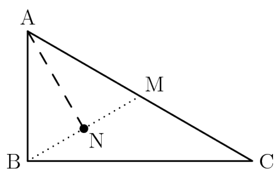
    Case 1: 4 doors are the same color: There are 4 ways to do this; one for each of the 4 colors.

    Case 2: 3 doors are the same color: There are 4 choices for which color we use three times and $\binom{4}{3}$ ways to choose the three doors that get painted the same color. Then, for the final door, which has already been decided, there are 3 colors we can use to paint it. Multiplying, there are $4 \times 4 \times 3 = 48$ ways for this case.

    In total there are 4^4 ways to paint the doors without any restrictions; subtracting what we don't, we find that there are $256 - (4 + 48) = \boxed{\textbf{(D) } 204}$ ways to do this.

11. Note that the songs must be in alternating order starting out with a jazz song, followed by non-jazz, followed by jazz, and so on. There are $5! = 120$ ways to arrange the jazz songs. The rest of the songs can occur in any order, so we have $4! = 24$ ways to do this, so our answer is $120 \cdot 24 = \boxed{\textbf{(C) } 2880}$.

12. Firstly, we see that AN is an altitude because triangle ABM is isosceles. Note that BN = MN = 2, and triangles ABN and AMN are both 30 - 60 - 90 triangles. Next, since BM = 4 = MC, triangle BMC is also isosceles. The measure of angle BMC is 120 ($180 - m\angle AMB$), and thus, $m\angle MBC = m\angle MCB = 30°$. Now, since triangle ABC is a 30 - 60 - 90 triangle, BC is $\boxed{\textbf{(E) } 4\sqrt{3}}$.



13. The probability that Tom gets cotton candy on 1/5 days is $(2/5) \cdot (3/5)^4 \cdot \binom{5}{1} = 810/3125$ because we need to choose one of the five days for him to get cotton candy. The probability that he gets cotton candy on 3/5 days is $(2/5)^3 \cdot (3/5)^2 \cdot \binom{5}{3} = 720/3125$. The probability he gets cotton candy on 5/5 days is $(2/5)^5 \cdot \binom{5}{5} = 32/3125$. Adding up the probabilities, we get $\boxed{\textbf{(C)}\ 1562/3125}$.

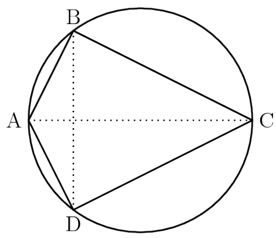
14. *Solution 1*: There are 4 multiples of three whose sum of digits is 3, 7 multiples whose sum of digits is 6, 10 multiples whose sum of digits is 9, 7 multiples whose sum of digits is 12, 4 multiples whose sum of digits is 15, and 1 multiple whose sum of digits is 18.

    To find the sum of the sums of the digits, we compute $(4 \times 3) + (7 \times 6) + (10 \times 9) + (7 \times 12) + (4 \times 15) + (1 \times 18) = \boxed{\textbf{(A)}\ 306}$.

    Solution 2: We compute (0+0)+(0+3)+(0+6)+(0+9)+(1+2)+...+(9+9). There are 34 terms in this sequence and the expected value of each term is $\frac{9}{2} + \frac{9}{2} = 9$. So the sum is $34 \times 6 = \boxed{\textbf{(A)}\ 306}$.

15. Because of the relationship between inscribed angles and arcs of a circle, $m\angle ADC = m\angle ABC = 90$. Thus, we can use the Pythagorean theorem on triangle ADC to compute the diagonal AC has length $3\sqrt{5}$. The area of this kite is the combined areas of triangles ADC and ABC, so the area of the kite is 18. We also know that the area of a kite is the product of its diagonals divided by 2. Thus, we can write this equation. $\frac{BD \times 3\sqrt{5}}{2} = 18$.

    Simplifying, we find that $BD = \boxed{\textbf{(B)}\ \dfrac{12\sqrt{5}}{5}}$.

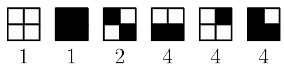


16. We note that triangle ABC is similar to triangle DEF. Thus, we can set up this equation: $\frac{y}{y+3} = \frac{2x}{x+6}$. Cross multiplying, we get the equation $xy + 6x = 2xy + 6x$. This reduces to $xy + 6x - 6y = 0$. Applying Simon's Favorite Factoring Trick, we can rewrite this equation as $(x-6)(y+6) = -36$. Then, testing the various factors of -36, we find that the (x, y) pairs that work are $(2,3), (3,6), (4,12)$, and $(5,30)$. Thus, there are $\boxed{\textbf{(E)}\ 4}$ pairs.

17. If their product is 10! and their LCM is 9!, then their GCD must be $10!/9! = 10$. If we divide both our numbers by 10 then we obtain a pair of relatively prime positive integers. We can thus establish a bijection between pairs of positive integers with product of 10! and GCD of 10 and relatively prime positive integers with product of $10!/(10^2) = 2^6 \times 3^3 \times 7$. If

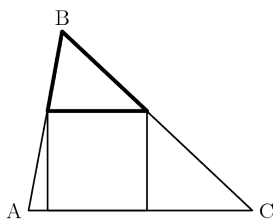
two numbers are relatively prime, then no number may divide both numbers, so we're simply distributing 2^6, 3^3, and 7 among our numbers. There are $2^3 = 8$ ways to distribute the powers of 2 among our numbers that are relatively prime, but we are counting pairs so order doesn't matter. Our answer is $8/2 = \boxed{\textbf{(C)}\ 4}$.

18. First note that there are $(2^4)^2 = 256$ ways to color both squares. Then, we find the possible distinct colorings, and the number of rotations that each of them can undergo. The six possible colorings, along with the number of rotations possible, is shown below:



Then, we use complementary counting to find non-distinct pairs of colorings. All we have to do is pair up colorings that only differ by a rotation. Thus, if a coloring had n rotations, the number of non-distinct pairings would be n^2. Applying this to all rotations, we have that the probability that the colorings are non-distinct is $\frac{1^2+1^2+2^2+4^2+4^2+4^2}{256} = \frac{27}{128}$. This is what we don't want, so we subtract this probability from one to get $1 - \frac{27}{128} = \boxed{\textbf{(E)}\ \frac{101}{128}}$.

19. We first find the side length of the first square, which is well-known to be $\frac{bh}{b+h} = \frac{20\cdot 16}{20+16} = \frac{80}{9}$. Thus ,the area of S_1 is simply $(\frac{80}{9})^2 = \frac{6400}{81}$. We then only focus on the triangle closest to vertex B, and let this be the new triangle:



Note that the bolded triangle (shown) and the original triangle, ΔABC, are similar by a factor of $\frac{\frac{80}{9}}{20} = \frac{4}{9}$, thus, the square's area will be multiplied by a factor of $(\frac{4}{9})^2 = \frac{16}{81}$. Continuing this use of similar triangles, we realize that the square's areas will continue to be multiplied by $\frac{16}{81}$.

Thus, the sum of the areas of all squares S_n is simply

$$(\frac{6400}{81}) + (\frac{6400}{81} \cdot \frac{16}{81}) + (\frac{6400}{81} \cdot (\frac{16}{81})^2) + (\frac{6400}{81} \cdot (\frac{16}{81})^3) + \dots$$

This sum is a geometric series, so our final answer is $\frac{\frac{6400}{81}}{1-\frac{16}{81}} = \boxed{\textbf{(C)}\ \frac{1280}{13}}$.

20. Each term of the sequence can be expressed as the form $n(n+1)(n+2)$. Expanding this, we get $n^3 + 3n^2 + 2n$. Now, we compute

$$(1^3 + 3 \times 1^2 + 2 \times 1) + (2^3 + 3 \times 2^2 + 2 \times 2) + \dots + (20^3 + 3 \times 20^2 + 2 \times 20)$$

This can be rewritten as

$$(1^3+2^3+3^3+...+20^3)+3(1^2+2^2+3^2+...+20^2)+2(1+2+3+...+20)$$

This simplifies to $(1+2+3+...+20)^2+3(\frac{20(20+1)(2\times 20+1)}{6})+2\binom{21}{2}=$ $\boxed{\textbf{(E) } 53130}$.

21. We note multiple, interesting things about this problem. We firstly note that, for given dimensions $a \times b$ and $y \times z$ of rectangles that Alpha and Omega choose, there is a unique configuration for these rectangles (as they must contain the corner-most unit squares). Also, for these given dimensions, their intersection will be of dimensions $(a+y-8)\times(b+z-8)$ by the Principle of Inclusion and Exclusion. Without loss of generality, assume that $a+y-8 < b+z-8$; we'll just have to multiply by two at the end. By noting that the factors of 8 are limited (the possible pairs are $1, 8$ and $2, 4$), we have that either

$$\begin{cases} a+y-8=1 & 0<a,y\leq 8 \\ b+z-8=8 & 0<b,z\leq 8 \end{cases} \text{ or } \begin{cases} a+y-8=2 & 0<a,y\leq 8 \\ b+z-8=4 & 0<b,z\leq 8 \end{cases}$$

After performing casework on a, y, b, z, we have that the first case has 8 solutions, and that the second case has 35 solutions. Adding and multiplying by two, we have $2\times(8+35)=$ $\boxed{\textbf{(A) } 86}$.

22. Note that, by Vieta's ,we have that $a=-(\alpha+\beta+\gamma)$, $b=\alpha\beta+\beta\gamma+\gamma\alpha$, and $c=-(\alpha\beta\gamma)$. The problem thus becomes

$$-(\alpha\beta\gamma-2\alpha\beta-2\beta\gamma-2\gamma\alpha+4\alpha+4\beta+4\gamma)=-2^{11}$$

for which we can (trivially) negate both sides. Using SFFT, we have

$$(\alpha-2)(\beta-2)(\gamma-2)=2^{11}-8=2^3\cdot 3\cdot 5\cdot 17$$

Thus, this problem boils down to a factoring problem, where $\alpha-2, \beta-2$, and $\gamma-2$ are factors of of the RHS. Noting that all of α, β, γ must be positive integers and must be in increasing order, we realize that the values of $\gamma-2$ are simply any factors greater than $\sqrt[3]{2^3\cdot 3\cdot 5\cdot 17}\approx 12.6$, except 15 (this is outnumbered by 17). Thus, we have that

$$\gamma \in \{17+2, 20+2, 21+2, 24+2, \dots, 680+2, 1020+2, 2040+2\}$$

Note that this set is simply the sum of all the factors of $2^{11}-8=2040$ except for factors less than 17 (this includes $1, 2, 3, 4, 5, 6, 8, 10, 12, 15$), each of which are increased by two. There are $4\cdot 2^3-10=22$ factors that satisfy, so the sum of all of them is thus

$$(1+2+4+8)(1+3)(1+5)(1+17)-(1+2+3+4+5+6+8+10+12+15)+2\cdot 22=$$

$\boxed{\textbf{(A) } 6458}$ and we are done. Phew!

23. We will attack this problem with linearity of expectation.

We have 4 cases:

1) The cubes that include the vertices of the box

Because each face of the box has a 1/3 probability of not being painted, so does each of the 3 possible faces of the cube that could potentially be painted. Therefore, the expected number of cubes that have no paint on them is $(1/3)^3 \times 8 = \frac{8}{27}$

2) The cubes that include exactly one edge

Same logic applies here. The probability that one of these cubes is not painted at all is $\frac{1}{3}^2$. There are a total of $4(1+2+3) = 24$ of these so the expected total from this is $\frac{24}{9} = \frac{8}{3}$.

3) The cubes that only include one face of the box

The probability that one of these not being painted is $\frac{1}{3}$. There are a total of $2(1 \times 2 + 1 \times 3 + 2 \times 3) = 22$ of these, so the expected total from this is $\frac{22}{3}$.

4) The cubes that have no potential of being painted (they are on the inside of the box).

There are $(3-2)(4-2)(5-2) = 6$ of these.

Our final expected total is $\frac{8}{27} + \frac{8}{3} + \frac{22}{3} + 6 = \frac{440}{27}$, which rounds down to $\boxed{\textbf{(B) } 16}$.

24. First, note that $\gcd(a^n, b^n) = \gcd(a, b)^n$. This can be proved directly. I'll leave it as an exercise to the reader to prove this.

Now, replace $\gcd(a, b)$ with g for simplicity. Then we have $g^3 - 18g^2 + 108g = 728$. We can rewrite this as

$$g^3 - 18g^2 + 108g - 216 = 512 \implies (g-6)^3 = 512$$

. Thus, the only solution that makes sense is 14. The remainder when 14^4 is divided by 10 is then $\boxed{\textbf{(D) } 6}$.

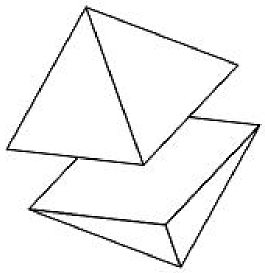
25. This is a two-part problem: find the radius of each of the spheres, and then find the volume common to each of the two types of solids.

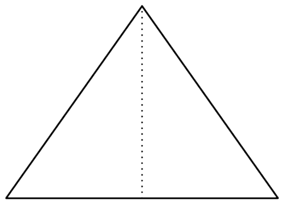
Step 1: Finding the Radius of Each Sphere

This is easy enough. Given the volume of a non-unit octahedron: we can try to instead find the volume of a unit octahedron, and then scale this up by a certain amount to get its side length for which we halve to get a sphere's radius. We note that an octahedron can be split into two right pyramids, as shown.

To determine the height of the pyramid, we take a cross section of one of these pyramids along one of its lateral edges, resulting in this figure:

The base is the diagonal of the square base, so its lengths must be $\sqrt{2}$. The dotted altitude is thus the height of the pyramid, and, by the Pythagorean Theorem, has length $\frac{\sqrt{2}}{2}$, so the volume of one of these pyramids is $\frac{1^2 \times \frac{\sqrt{2}}{2}}{3} = \frac{\sqrt{2}}{6}$, and the volume of the unit octahedron is $\frac{\sqrt{2}}{6} \times 2 = \frac{\sqrt{2}}{3}$. The volume





of our given octahedron is 2, so the volume must have been scaled up by a factor of $3\sqrt{2} = \sqrt{18}$, and the side lengths of it must be $\sqrt[3]{\sqrt{18}} = \sqrt[6]{18}$.
■

Step 2: Finding the Volume of the Intersection

Reconsider our square pyramid demonstrated in the first step.

We break it up into cases, which we will add to receive the full volume.

Case 1: The 4 spheres that contain vertices from the square base.

In a square, each angle is 90 degrees. This is $\frac{1}{4}$ of 360. Also note that the angle formed by an edge from the square and an edge not from the square is 60 (b/c equilateral triangles). This is $\frac{1}{6}$ of 360. So this means that the volume of one of these is $\frac{1}{24}$ the volume of the sphere. There are 4 of these, altogether in the pyramid, so their combined volume is $\frac{1}{6}$ of a sphere.

Case 2: The sphere that includes an apex.

The 2 angles in this case are 60 and 60, and $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, so the volume of one of these is $\frac{1}{36}$ of the sphere.

Summing up, we get that $2(\frac{1}{6} + \frac{1}{36}) = \frac{7}{18}$ of the sphere is their combined volume.

The volume of the sphere is $\frac{4}{3}(\sqrt[6]{18})^3\pi = 8\sqrt{2}\pi$, and $\frac{7}{18}$ of that volume is $\frac{7}{18} \times 8\sqrt{2}\pi = \boxed{\textbf{(D)}\ \frac{14\sqrt{2}\pi}{9}}$, and we are done.