MPHIL REPORT

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Abstract

Some definitions and concepts are studied,Manifolds, diffeomorphism functions, Tangent space, Riemmannian manifolds and Riemannian metric. Also, the Kahler geometry of Lu-Page-Pope quasi-Einstein Metrics on $\mathbb{CP}^2\sharp\mathbb{CP}^2$ is studied .

1 Foundational Material

1.1 manifold

1.1.1 Definition of manifold

A $manifold$ M of dimension d is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomoorphic to an open subset Ω of $\mathbb{R}^d$ Such a homeomorphism $x : U \to \Omega$ is called a ($coordinate$) chart.An atlas is a family of charts $\{U_\alpha, x_\alpha\}$ for which the U_α constitute an open covering of M. Note that A point $p \in U_\alpha$ is determined by $x_\alpha(p)$; hence it is often identified with $x_\alpha(p)$.Often, also the index α is omitted, and the components of $x(p) \in \mathbb{R}^d$ are called local coordinates of p.

1.1.2 differentiable atlas

An atlas $\{U_\alpha, x_\alpha\}$ on called a manifold is called diffeomorphism if all chart transitions $x_\beta \circ x_\alpha : x_\alpha(U_\alpha \bigcap U_\beta) \to x_\beta(U_\alpha \bigcap U_\beta)$ are differentiable of class C^∞ (in case $U_\alpha \bigcap U_\beta \neq \phi$). A maximal differentiable atlas is called a differentiable structure, and a differentiable manifold of dimension d is a manifold of dimension d with a differentiable structure. From now on, all atlases are supposed to be differentiable. Two atlases are called compatible if their union is again an atlas. In general, a chart is called compatible with an atlas if adding the chart to the atlas yields again an atlas. An atlas is called maximal if any chart compatible with it is already contained in it.

1.1.3 examples

1.2 Tangent Space

1.2.1 Tangen vector and Tangent Space

a tangent vector is an infinitesimal displacement at a specific point on a manifold. The set of tangent vectors at a point P forms a vector space called the

tangent space at P, and the collection of tangent spaces on a manifold forms a vector bundle called the tangent bundle.
One of the most important tools in the theory of smooth manifolds is the notion of the tangent space to a manifold at a given point. In section five David Mond (Lecture Note) have presented three different approaches to the defnition of tangent spaces and we want to explain why they are equivalent. All these approaches are useful because each one has some properties. The "tangent space" (whatever that means) to a smooth manifold M at a point $p \in M$ is denoted T_pM it is equipped with a canonical structure of a real vector space whose dimension is equal to the dimension, dim_pM, of M at p. As a set, T_pM consists of all " tangent vectors" to M at p, so understanding the defnition of T_pM is more or less the same as understanding the defnition of a tangent vector Let $U \subset \mathbb{R}^n$ be an open set (where$n \geq 1$) ,and fix $p \in U$. Whatever the definition of a tangent space is, the answer it should give in this particular case is that T_pU is the same vector space $\mathbb{R}^n$ then how can we describe this vector space in terms of the natural smooth manifold structure that we have on U, without explicitly referring to the embedding $U \hookrightarrow \mathbb{R}^n$
There are three difinitions of tangent space in David Mond's lecutre note which can be classified into:
via derivations acting on function
A dreivation at p is an $\mathbb{R}$ lineaner map $C^\infty(M) \to \mathbb{R}$ which is aderivation in Leibnit's sense: $f(x)v \cdot g + g(x)v \cdot f$ .The tangent space to M at p is the space of derivations at p.
the coordinates were not used to define it. the basis of definition 1.1 is that if $f \in C^\infty(M)$ then $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ can be written as $f(x^1, ..., x^n)$. The derivation $\frac{\partial}{\partial x^i}$ is the usual partial derivative :$\frac{\partial}{\partial x^i}[f] = \frac{\partial f}{\partial x^i}$
via coordinates
Let $\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$ be all charts containing p, so that $p \in U_\alpha$. For each such chart introduce a copy of $\mathbb{R}^n$, denoted by $\mathbb{R}^n_\alpha$, and form the disjoint union of all these vector spaces. :

$$\coprod_\alpha \mathbb{R}^n_\alpha$$

Define an equivalence relation $\sim$ on this disjoint union by declaring $v_\alpha \in \mathbb{R}^n_\alpha$ and $v_\beta \in \mathbb{R}^n_\beta$ to be equivalent:
$v_\alpha \sim v_\beta$ if and only if $d_{\phi_\alpha(p)}(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))v_\alpha$. Then define $T_pM = \coprod_\alpha \mathbb{R}^n_\alpha \diagup \sim$.
via curves
A smooth curve γ through p is a map $\gamma(-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$. Two smooth curves γ_1, γ_2 through p "agree to first order" if, in some coordinate chart containing p their coordinate images agree to first order as curves in $\mathbb{R}^n$. A tangent vector at p is an equivalence class of curves passing through p. The tangent space is the set of all such equivalence classes of curves.This definition has a clear geometric meaning and is fairly easy to state. It also makes the defnition of the diferential of a smooth map very easy and natural, and it makes the chain rule completely trivial.
Two main disadvantages of the above approach to defning T_pM are that it doesn't explain where the vector space structure comes from, and it does not generalize well to algebraic varieties.
A tangent vector at a point P on a manifold is a tangent vector at P in a co-

ordinate chart. A change in coordinates near P causes an invertible linear map of the tangent vector's representations in the coordinates. This transformation is given by the Jacobian, which must be nonsingular in a change of coordinates. Hence the tangent vectors at P are well-defined. A vector field is an assignment of a tangent vector for each point. The collection of tangent vectors forms the tangent bundle, and a vector field is a section of this bundle.
Let V be a finite dimensional vector spcae and $\{\overrightarrow{e}_1, ..., \overrightarrow{e}_n\}$ be a basis for $V, j \in \{1, ..., n\}$ then we can define a dual vector space V^* such that $\theta_i \in V^*$ a basis of V^* and $F : V \to \mathbb{R}$ is linear
Since we have already roted that $dim(V) = dim(V^*)$ to prove that $\{\theta_1, \theta_2, .., \theta_n\}$ is a basis for V^*, it suffices to show that $span(\theta_1, \theta_2, .., \theta_n) = V^*$. Along the way we will verity the statment that if $F \in V^*$ then $F = \sum_{i=1}^n F(\overrightarrow{e}_i)\theta_i$
Let $F \in V^*$ be arbitrary. To see that $F = \sum_{i=1}^n F(\overrightarrow{e}_i)\theta_i$ it suffices to show that $F(\overrightarrow{v}) = \sum_{i=1}^n F(\overrightarrow{e}_i)\theta_i(\overrightarrow{v})$ for all $\overrightarrow{v} \in V$
To begin let $\overrightarrow{v} \in V$, since $\{\overrightarrow{e}_1, ..., \overrightarrow{e}_n\}$ is a basis for V then there exist $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $\overrightarrow{v} = \sum_{i=j}^n \lambda_j \overrightarrow{e}_j$
Now $\sum_{i=1}^n F(\overrightarrow{e}_i)\theta_i(\overrightarrow{v}) = \sum_{i=1}^n F(\overrightarrow{e}_i)\theta_i(\sum_{j=1}^n \lambda_j \overrightarrow{e}_j)$ but as we know that $\theta_i(\overrightarrow{v}) = \theta_i(\lambda_1 \overrightarrow{v}_1 + ... + \lambda_n \overrightarrow{v}_n) = \lambda_1\theta_i(\overrightarrow{v}_1) + ... + \lambda_i\theta_i(\overrightarrow{v}_i) + .. + \lambda_n\theta_n(\overrightarrow{v}_n) = \lambda_1.0 + ... + \lambda_i.1 + ... + \lambda_n.0 = \lambda_i$ because $\theta_i(\overrightarrow{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
Then $\sum_{i=1}^n F(\overrightarrow{e}_i)\theta_i(\overrightarrow{v}) = \sum_{i=1}^n F(\overrightarrow{e}_i)\lambda_i = \sum_{i=1}^n \lambda_i F(\overrightarrow{e}_i) = F(\sum_{j=1}^n \lambda_j(\overrightarrow{e}_j) = F(\overrightarrow{v})$

1.3 Riemannian Geometry

1.3.1 Riemannnian Manifold

The Riemannian metric g provides us with an inner product on each tangent space and can be used to measure angles and the lengths of curves in the manifold. This defnes a distance function and turns the manifold into a metric space in a natural way. The Riemannian metric on a differentiable manifold is an important example of what is called a tensor field.

1.3.2 examples

1.3.3 connection

1.3.4 curvature and Ricci tensor

2 Introduction

2.1 Almost complex Structure

2.2 Hermitian Almost complex Manifold

A quasi-Einstein metric is a complete Riemannian manifold (M, g) where the metric g satisfies:

$$Ric(g) + \nabla^2\phi - \frac{1}{m}d\phi \otimes d\phi = \lambda g$$

3 $U(2)$- $INVARINTK\ddot{A}LERMETRICON$ $\mathbb{CP}^2\sharp\mathbb{CP}^2$

4 EXPLICIT METRICS

4.1 Page's Einstein metric

4.2 THE Koiso-Cao Kahler Ricci Solution

4.3 THE Lu-Page-Pope Metrics

4.4 Kim-Kim first integral

4.5 Boundary behavior

5 Attempt at an integral constraint

5.1 Wang-Zhu

6 References