Krylov Subspace Methods in Model Order Reduction

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Outline

Introduction

Model Reduction Problem Revisited

Given a MIMO state space model

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{1}$$

where, $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{p\times n}$,
$\mathbf{u} \in \mathbb{R}^{m}, \mathbf{y} \in \mathbb{R}^{p}, \mathbf{x} \in \mathbb{R}^{n}$ and n is sufficiently large.

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$\mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p, \mathbf{x} \in \mathbb{R}^n$ and n is sufficiently large.
It is required to obtain the following reduced order model

$$\begin{aligned}\mathbf{E_r}\dot{\mathbf{z}} &= \mathbf{A_r}\mathbf{z} + \mathbf{B_r}\mathbf{u} \\ \mathbf{y} &= \mathbf{C_r}\mathbf{z}\end{aligned} \tag{2}$$

where, $\mathbf{E_r}, \mathbf{A_r} \in \mathbb{R}^{q\times q}, \mathbf{B_r} \in \mathbb{R}^{q\times m}, \mathbf{C_r} \in \mathbb{R}^{p\times q}$,
$\mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p, \mathbf{z} \in \mathbb{R}^q \quad \boldsymbol{q << n}$
$\mathbf{E_r} = \mathbf{W}^\mathsf{T}\mathbf{E}\mathbf{V}, \mathbf{A_r} = \mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}, \mathbf{B_r} = \mathbf{W}^\mathsf{T}\mathbf{B}, \mathbf{C_r}^\mathsf{T} = \mathbf{C}^\mathsf{T}\mathbf{V}$

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$$\begin{aligned} \mathbf{E\dot{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{1}$$

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$\mathbf{W}, \mathbf{V}$ are suitable Krylov subspace based projection matrices.

Moments and Markov Parameters

The transfer function of the system in (1) is

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-\mathbf{1}}\mathbf{B}$$

By assuming that $\mathbf{A}$ is nonsingular, the Taylor series of this transfer function around zero is:

$$\mathbf{G}(s) = -\mathbf{C}\mathbf{A}^{-\mathbf{1}}\mathbf{B} - \mathbf{C}(\mathbf{A}^{-\mathbf{1}}\mathbf{E})\mathbf{A}^{-\mathbf{1}}\mathbf{B}s - \cdots - \mathbf{C}(\mathbf{A}^{-\mathbf{1}}\mathbf{E})^{i}\mathbf{A}^{-\mathbf{1}}\mathbf{B}s^{i} - \cdots$$

Coefficients of powers of s are known as moments
i-th moment:

$$\mathbf{M}_i^0 = \mathbf{C}(\mathbf{A}^{-\mathbf{1}}\mathbf{E})^{\mathbf{i}}\mathbf{A}^{-\mathbf{1}}\mathbf{B}, \quad i = 0, 1, \ldots$$

Also,

$$\mathbf{M}_i^0 = -\frac{1}{i}\frac{d^i\mathbf{G}(s)}{ds^i}\Big|_{s=0}$$

is the value of subsequent derivatives of the transfer function $\mathbf{G}(s)$ at the point $s = 0$

Contd...

A different series in terms of negative powers of s is obtained when expanded about $s \to \infty$

$$\mathbf{G}(s) = \mathbf{C}\mathbf{E}^{-1}\mathbf{B}s^{-1} + \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})\mathbf{E}^{-1}\mathbf{B}s^{-2} + \cdots + \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{\mathbf{i}}\mathbf{E}^{-1}\mathbf{B}s^{-i} + \cdots$$

and the coefficients are known as Markov parameters.

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- Model reduction is achieved by the means of matching of Moments (Markov parameters)
- Explicit moment matching becomes numerically cumbersome for large system order
- Go for *implicit* moment matching: Krylov subspace based Projection

Remarks 1

- Asymptotic Waveform Evaluation (AWE) method is based on explicit moment matching
- Matching at $s = 0$ is known as Padé Approximation, and steady state response (low frequency) is reflected in the reduced order model.
- Matching at $s \to \infty$ is known as Partial Realization, and the reduced order model is a good approximation of the HF response.
- Matching at $s = s_0$, i. e. at some arbitrary value of s is known as Rational Interpolation and is aimed at approximating system response at specific frequency band of interest.

Defining Krylov Subspace

$$\mathcal{K}_q(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \dots, \mathbf{A}^{\mathbf{q-1}}\mathbf{b}\},$$

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$ is called the starting vector. q is some given positive integer called index of the Krylov sequence.

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- The vectors $\mathbf{b}, \mathbf{Ab}, \dots$, constructing the subspace are called basic vectors.

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- The Krylov subspace is also known as *controllability* subspace in control community.
- For each state space, there are two Krylov subspaces that are dual to each other, input Krylov subspace and output Krylov subspace.
- Either or both of subspaces can be used as projection matrices for model reduction.

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- The respective method is then called One-Sided/Two-sided

Input and Output Krylov Subspaces

Input Krylov subspace

$$\mathcal{K}_{q_1}\left(\mathbf{A^{-1}E}, \mathbf{A^{-1}b}\right) = \text{span}\left\{\mathbf{A^{-1}b}, \dots, \left(\mathbf{A^{-1}E}\right)^{q_1-1}\mathbf{A^{-1}b}\right\}$$

Output Krylov Subspace

$$\mathcal{K}_{q_2}\left(\mathbf{A^{-T}E^{T}}, \mathbf{A^{-T}c}\right) = \text{span}\left\{\mathbf{A^{-T}c}, \dots, \left(\mathbf{A^{-T}E^{T}}\right)^{q_2-1}\mathbf{A^{-T}c}\right\}$$

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$\mathbf{V}$ is any basis of Input Krylov Subspace
$\mathbf{W}$ is any basis of Output Krylov Subspace

Moment Matching: SISO

Theorem If the matrix $\mathbf{V}$ used in (2), is a basis of Krylov subspace $\mathcal{K}_{q_1}\left(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b}\right)$ with rank q and matrix $\mathbf{W}$ is chosen such that the matrix $\mathbf{A}_r$ is nonsingular, then the first q moments (around zero) of the original and reduced order systems match.

Moment Matching: SISO

Proof: The zero-th moment of the reduced system is

$$m_{r0} = \mathbf{c}_r^\mathsf{T}\mathbf{A}_r^{-1}\mathbf{b}_r = \mathbf{c}^\mathsf{T}\mathbf{V}\left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-\mathbf{1}}\mathbf{W}^\mathsf{T}\mathbf{b}$$

The vector $\mathbf{A}^{-\mathbf{1}}\mathbf{b}$ is in the Krylov subspace and it can be written as a linear combination of the columns of the matrix $\mathbf{V}$,

$$\exists \mathbf{r}_0 \in \mathbb{R}^q : \mathbf{A}^{-\mathbf{1}}\mathbf{b} = \mathbf{V}\mathbf{r}_0$$

Therefore,

$$\left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^\mathsf{T}\mathbf{b} = \left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^\mathsf{T}\left(\mathbf{A}\mathbf{A}^{-1}\right)\mathbf{b}$$

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Therefore,

$$\begin{aligned} \left(\mathbf{W}^\mathsf{T} \mathbf{A} \mathbf{V} \right)^{-1} \mathbf{W}^\mathsf{T} \mathbf{b} &= \left(\mathbf{W}^\mathsf{T} \mathbf{A} \mathbf{V} \right)^{-1} \mathbf{W}^\mathsf{T} \left(\mathbf{A} \mathbf{A}^{-1} \right) \mathbf{b} \\ &= \left(\mathbf{W}^\mathsf{T} \mathbf{A} \mathbf{V} \right)^{-1} \mathbf{W}^\mathsf{T} \mathbf{A} \mathbf{V} \mathbf{r}_0 \end{aligned}$$

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Contd...

With this, m_{r0} becomes

$$m_{r0} = \mathbf{c}^\mathsf{T}\mathbf{V}\left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-\mathbf{1}}\mathbf{W}^\mathsf{T}\mathbf{b} = \mathbf{c}^\mathsf{T}\mathbf{V}\mathbf{r}_0 = \mathbf{c}^\mathsf{T}\mathbf{A}^{-1}\mathbf{b} = m_0$$

For the next moment (first moment) consider the following result:

$$\begin{aligned}\left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^\mathsf{T}\mathbf{E}\mathbf{V}\left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-1}\left(\mathbf{W}^\mathsf{T}\mathbf{b}\right) &= \left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^\mathsf{T}\mathbf{E}\mathbf{V}\mathbf{r}_0 \\ &= \left(\mathbf{W}^\mathsf{T}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^\mathsf{T}\mathbf{E}\mathbf{A}^{-1}\mathbf{b}\end{aligned}$$

and the fact that $\mathbf{A}^{-\mathbf{1}}\mathbf{E}\mathbf{A}^{-\mathbf{1}}\mathbf{b}$ is also in the Krylov subspace can be written as $\mathbf{A}^{-\mathbf{1}}\mathbf{E}\mathbf{A}^{-\mathbf{1}}\mathbf{b} = \mathbf{V}\mathbf{r}_1$

Contd...

Thus,

$$\left(\mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^{\mathsf{T}}\left(\mathbf{A}\mathbf{A}^{-1}\right)\mathbf{E}\mathbf{A}^{-\mathbf{1}}\mathbf{b} = \left(\mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}\mathbf{r}_1$$

Contd...

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$$m_{r1} = \mathbf{c}^{\mathsf{T}}\mathbf{V}\left(\mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^{\mathsf{T}}\mathbf{E}\mathbf{V}\left(\mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{W}^{\mathsf{T}}\mathbf{b}$$

Contd...

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$$= \mathbf{c}^{\mathsf{T}}\mathbf{V}\mathbf{r}_{\mathbf{1}} = \mathbf{c}^{\mathsf{T}}\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{b}$$

Contd...

Thus,

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Remarks 2

- For the second moment, the results of first moment can be used and the fact that $\left(\mathbf{A^{-1}E}\right)^2 \mathbf{A^{-1}b}$ can be written as a linear combination of columns of matrix $\mathbf{V}$
- The proof can be continued by repeating these steps (Induction) until $m_{r(q-1)} = m_{(q-1)}$ i.e. q moments match.
- The method discussed above was one-sided as we did not go for computing $\mathbf{W}$. Usually, $\mathbf{W} = \mathbf{V}$ is chosen
- In two-sided method $\mathbf{W}$ is chosen to be the basis of output Krylov subspace, then $2q$ moments can be matched.
- Proof is similar for matching Markov parameters and the MIMO case [3,4].

Issues with Krylov Methods

Major issues with Krylov Subspace based MOR Methods:

1. Orthogonalization
2. Stopping Point of Iterative Scheme

Orthogonalization

- The Krylov vectors are known to lose independence readily and tend to align towards the dominant vector, even for moderate values of n and q.

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- However, classical GS is also known to be unstable
- Go for Modified GS methods — **Arnoldi (Unsymmetric $\mathbf{A}$) / Lanczos (Symmetric $\mathbf{A}$)**

Arnoldi Algorithm

Using Modified Gram-Schmidt Orthogonalization

Algorithm 1 Arnoldi

Start: Choose initial starting vector $\mathbf{b}, \mathbf{v} = \frac{\mathbf{b}}{\|\mathbf{b}\|}$
Calculate the next vector: $\hat{\mathbf{v}}_i = \mathbf{A}\mathbf{v}_{i-1}$
Orthogonalization:
for $j = 1$ to $i - 1$ **do**
    $\mathbf{h} = \hat{\mathbf{v}}_i^{\mathsf{T}}\mathbf{v}_j,\ \hat{\mathbf{v}}_\mathbf{i} = \hat{\mathbf{v}}_\mathbf{i} - \mathbf{h}\mathbf{v}_\mathbf{j}$
    Normalization:
    i-th column of $\mathbf{V}$ is $\mathbf{v}_i = \frac{\hat{\mathbf{v}}_i}{\|\hat{\mathbf{v}}_i\|}$ stop if $\hat{\mathbf{v}}_i = 0$
end for

Output of Arnoldi Iteration:

1. Orthonormal Projection matrix $\mathbf{V}$,
2. Hessenberg Matrix $\mathbf{H} = \mathbf{V}^{\mathsf{T}}\mathbf{A}\mathbf{V}$

Stopping Criterion

- When to stop the iterative scheme?
  is another question to be answered
- This also decides the size of the ROM
- TU-M: Singular values based stopping criterion.[1]
- IIT-D: A more efficient criterion based on a index known as CNRI[2] is proposed.[3]

[1] B. Salimbahrami and Lohmann, B., "Stopping Criterion in Order Reduction of Large Scale Systems Using Krylov Subspace Methods", Proc. Appl. Math. Mech., 4: 682–683, 2004.

[2] Coefficent of Numerical Rank Improvement

[3] M. A. Bazaz, M. Nabi and S. Janardhanan. "A stopping criterion for Krylov-subspace based model order reduction techniques". Proc. Int. Conf. Modelling, Identification & Control (ICMIC), pp. 921 - 925, 2012

Comparison with Balanced Truncation

Parameter	BT	Krylov
No. of Flops	$\mathcal{O}\left(n^3\right)$	$\mathcal{O}\left(q^2 n\right)$
Numerical Reliability for large n	No	Yes
Accuracy of the reduced system	More Accurate	Less Accurate
Range of Applicability	$\sim 10^3$	$\sim 10^4$ or higher
Stability Preservation	Yes	No
Iterative Method	No	Yes
Reliable Stopping Criterion	Yes	No*

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5. Zhaojun Bai, Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems, *Applied Numerical Mathematics*, 43 (2002), pp 9-44.

Thanks!

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