Introduction to Cartesian Coordinates in Geometry

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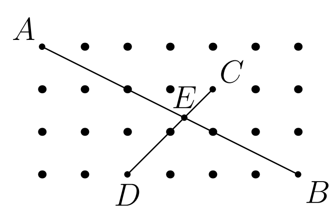
1 Introduction

The technique of coordinates in geometry is a very valuable one. Coordinates are applicable to many problems in geometry. The basic method is to put the diagram on a coordinate plane and use the distance formula, midpoint formula, systems of equations, shoelace formula etc. and bash the problem. While some of these problems may have a simpler synthetic solution, they are all approachable with the method of coordinates.

2 Examples

1. The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE. (AMC 10)



Solution: While this problem can also be solved through similar triangles, coordinates are the most obvious approach to the problem (we are already given a coordinate system). Let point A be $(0, 3)$, point B be $(6, 0)$, point C be $(4, 2)$, and point D be $(2, 0)$ (Note that the bottom left point is $(0, 0)$). To find the coordinates of point E, we need to find the equations of lines AB and DC. We find that line AB is defined by the equation $y = (\frac{-1}{2})x + 3$ and that line DC is defined by the equation $y = x - 2$. We can solve this system of equations to find the intersection, point E! We substitute $x - 2$ in for y in the first equation, and solving, we find that the intersection point is $E(\frac{10}{3}, \frac{4}{3})$. Using the distance formula, we find that $AE = 5\sqrt{5}/3$.

2. Consider a rectangle $ABCD$. Let M be a point on the segment AB such that $AM = 8$ and $MB = 12$. Let N be a point on the segment BC such that $BN = 4$ and $NC = 8$. Let P be a point on the segment CD such that $CP = 8$ and $PD = 12$. Let Q be a point on the segment AD such that $DQ = 4$ and $QA = 8$. Let O be the point of intersection of MP and NQ. Find the area of the quadrilateral $MONB$. (Mathcounts)

Solution: Let $B = (0, 0)$, $A = (0, 20)$, $D = (12, 20)$, and $C = (12, 0)$. Therefore, we can label all the other points: $M = (0, 12)$, $Q = (8, 20)$, $P = (12, 8)$ and $N = (4, 0)$. To find the coordinates of point O, we again can set up two equations and solve a system. The equation of MP is $y = \frac{-1}{3}x + 12$ and the equation of QN is $y = 5x - 20$. Since point O satisfies both these equations, we solve the system to get $O = (6, 10)$ (Note that we could have used symmetry to get this too). Now we can find the area of $MONB$ by using either the Shoelace Formula, or by drawing a segment parallel to BC from O to AB and drawing a segment parallel to AB from N to the other segment, creating two triangles and a rectangle. Either way, our answer is 56.

3 Exercises

(Note: All exercises are meant to be solved through the use of cartesian coordinates)

1. The legs of right triangle ABC have lengths 10 and 24, with $AB = 10$ and $BC = 24$. If AD and CE are medians that intersect at point F, find $[FBC]$. (note: $[FBC]$ denotes the area of $[FBC]$)

2. Point B lies on line segment $\overline{AC}$ with $AB = 16$ and $BC = 4$. Points D and E lie on the same side of line AC forming equilateral triangles $\triangle ABD$ and $\triangle BCE$. Let M be the midpoint of $\overline{AE}$, and N be the midpoint of $\overline{CD}$. The area of $\triangle BMN$ is x. Find x^2. (AIME)

4 Solutions

1. Let $B = (0, 0)$, $A = (0, 10)$, and $C = (24, 0)$. Thus, $E = (0, 5)$ and $D = (12, 0)$. We can set up a system of equations to find that $F = (8, \frac{10}{3})$. We can either drop an altitude of length $\frac{10}{3}$ to BC, or use Shoelace. Either way, $[FBC] = 40$.

2. Set point A as $(0, 0)$, point B as $(16, 0)$, and point C as $(20, 0)$. Using $30 - 60 - 90$ and equilateral triangle calculations, point D is $(8, 8\sqrt{3})$ and point E is $(18, 2\sqrt{3})$. Finding the midpoint of AE and CD gives us M at point $(9, \sqrt{3})$ and N at $(14, 4\sqrt{3})$. Finally, we can use the Pythagorean Theorem to find that $BM = MN = BN = 2\sqrt{13}$. Using the equilateral triangle formula gives us $x = 13\sqrt{3}$, so $x^2 = 507$.